TROPICALIZATION AND IRREDUCIBILITY OF GENERALIZED VANDERMONDE DETERMINANTS

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ABSTRACT. We find geometric and arithmetic conditions in order to characterize the irreducibility of the determinant of the generic Vandermonde matrix over the algebraic closure of any field k. We also characterize those determinants whose tropicalization with respect to the variables of a row is irreducible.

1. INTRODUCTION

Let n, N positive integers, $\mathbf{X}_1, \ldots, \mathbf{X}_N$ n-tuples of indeterminates, i.e.

$$\mathbf{X}_i := \big(X_{i1}, \ldots, X_{in}\big) \quad i = 1, \ldots, N$$

where each X_{ij} is an indeterminate, and $\Gamma := \big(\gamma_1, \ldots, \gamma_N\big)$, a N-tuple of vectors in $\mathbb{N}^n$, $\gamma_j = (\gamma_{j1}, \ldots, \gamma_{jn})$. Set

$$V(\underline{\mathbf{X}}, \Gamma) := \begin{vmatrix} X_{11}^{\gamma_{11}} X_{12}^{\gamma_{12}} \cdots X_{1n}^{\gamma_{1n}} & \ldots\ldots & X_{11}^{\gamma_{N1}} X_{12}^{\gamma_{N2}} \cdots X_{1n}^{\gamma_{Nn}} \\ X_{21}^{\gamma_{11}} X_{22}^{\gamma_{12}} \cdots X_{2n}^{\gamma_{1n}} & \ldots\ldots & X_{21}^{\gamma_{N1}} X_{22}^{\gamma_{N2}} \cdots X_{2n}^{\gamma_{Nn}} \\ & \ldots\ldots\ldots & \\ X_{N1}^{\gamma_{11}} X_{N2}^{\gamma_{12}} \cdots X_{Nn}^{\gamma_{1n}} & \ldots\ldots & X_{N1}^{\gamma_{N1}} X_{N2}^{\gamma_{N2}} \cdots X_{Nn}^{\gamma_{Nn}} \end{vmatrix}$$

We call the polynomial $V(\underline{\mathbf{X}}, \Gamma) \in \mathbb{Z}\big[(X_{i,j})_{1 \le i \le N, 1 \le j \le n}\big]$ the *Generalized Vandermonde determinant* associated to Γ.

Example 1.1. If $n = 1$ and $\Gamma = (0, 1, \ldots, N-1)$, then

$$V(\underline{\mathbf{X}}, \Gamma) = \pm \prod_{1 \le i < j \le N} (X_{i1} - X_{j1}),$$

the classical Vandermonde determinant.

Example 1.2. It is a classical result (see for instance [Mac]) that if $n = 1$, then for any set $\Gamma \subset \mathbb{N}$ of N elements, the determinant $V(\underline{\mathbf{X}}, \Gamma)$ is a multiple of the classical Vandermonde determinant $\prod_{1 \le i < j \le N} (X_{i1} - X_{j1})$.

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Example 1.3. Suppose $n=2$, $N=3$ and $\Gamma := ((2,0),(0,2),(2,2))$.

By computing the 3×3 determinant we have that $V(\underline{\mathbf{X}},\Gamma)$ is equal to

$$X_{11}^2X_{22}^2X_{31}^2X_{32}^2 - X_{11}^2X_{32}^2X_{21}^2X_{22}^2 - X_{12}^2X_{21}^2X_{31}^2X_{32}^2$$
$$+X_{12}^2X_{31}^2X_{21}^2X_{22}^2 + X_{11}^2X_{12}^2X_{21}^2X_{32}^2 - X_{11}^2X_{12}^2X_{31}^2X_{22}^2.$$

Set $\Gamma' := \{(1,0),(0,1),(1,1)\}$, it is easy to see that

- If $\mathrm{char}(k)\neq 2$, then $V(\underline{\mathbf{X}},\Gamma)$ is absolutely irreducible over $k\big[(X_{i,j})_{1\le i\le N, 1\le j\le n}\big]$ (i.e. irreducible over $\overline{k}\big[(X_{i,j})_{1\le i\le N, 1\le j\le n}\big]$, $\overline{k}$ being the algebraic closure of k).
- If $\mathrm{char}(k)=2$, then $V(\underline{\mathbf{X}},\Gamma) = V(\underline{\mathbf{X}},\Gamma')^2$ in $k\big[(X_{i,j})_{1\le i\le N, 1\le j\le n}\big]$.

Example 1.4. As an easy exercise, it can be proved that if $\Gamma\subset\mathbb{N}^n$ is contained in an affine line, then $V(\underline{\mathbf{X}},\Gamma)$ factorizes in a similar way as in the Example 1.2.

In the univariate case ($n=1$), the Vandermonde determinant is associated with the interpolation problem, and it has been extensively studied (see [GV, Gow, ElM] and the references therein). The multivariate interpolation problem is naturally associated with Generalized Vandermonde determinants, and there is also an extensive and current literature on the topic. See for instance [CL, GS, LS, Olv, Zhu].

The purpose of this article is to study the irreducibility of $V(\underline{\mathbf{X}},\Gamma)$. As Example 1.3 suggests, the answer will depend on the characteristic of k. Also, our intuition with the univariate case may lead us to believe that Generalized Vandermonde determinants have lots of irreducible factors. Our main result essentially tell us that in general these polynomials are absolutely irreducible.

There are some trivial factors that can be already read from the set of exponents. Let $\overline{\gamma} := (g_1\ldots,g_n)$ where each g_i is defined as $\min\{\gamma_{1i},\ldots,\gamma_{Ni}\}$, $i=1,\ldots,n$. It is easy to check that the following factorization holds:

$$V\big(\underline{\mathbf{X}},(\gamma_1,\ldots,\gamma_N)\big) = \prod_{i=1}^{n}\left(\prod_{j=1}^{N}X_{ij}\right)^{g_i} V\big(\underline{\mathbf{X}},(\gamma_1-\overline{\gamma},\ldots,\gamma_N-\overline{\gamma})\big),$$

and $V\big(\underline{\mathbf{X}},(\gamma_1-\overline{\gamma},\ldots,\gamma_N-\overline{\gamma})\big)$ has no monomial factor. Let d_Γ be the largest integer such that $\frac{1}{d_\Gamma}\{\gamma_1-\overline{\gamma},\ldots,\gamma_N-\overline{\gamma}\}\subset\mathbb{N}^n$, and $\mathcal{L}_\Gamma\subset\mathbb{R}^n$ the affine subspace spanned by Γ.

Theorem 1.5. *Let $N\ge 3$. The Vandermonde polynomial $V(\underline{\mathbf{X}},\Gamma)$ is irreducible in $\overline{k}\big[(X_{ij})_{1\le i,j,\le N}\big]$ if and only if the following three conditions apply:*

- $\dim(\mathcal{L}_\Gamma)\ge 2$,
- $\gcd\big(\mathbf{X}^{\gamma_i}\big)_{1\le i\le N} = 1$, *equivalently* $\overline{\gamma}=(0,\ldots,0)$,
- *$\mathrm{char}(k)$ does not divide d_Γ.*

Note that $\dim(\mathcal{L}_\Gamma)\ge 2$ implies $N\ge 3$ and $n>1$. When $n=2$ and $N=3,4$, Theorem 1.5 can also be obtained from an application of Ostrowski's work [Ost] on the irreducibility of fewnomials (see also [BP]). We will prove the general case by making use of Bertini's Theorem on the variety defined by $V(\underline{\mathbf{X}},\Gamma)$, and applying some results concerning algebraic independence of maximal Vandermonde minors gotten in [Tab].

When dealing with the problem of factorizing multivariate polynomials, several approaches like those given in [Ost, Gao] focus on the irreducibility of the Newton polytope of the polynomial with respect to the operation of computing Minkowski sums, which gives sufficient conditions to show irreducibility. A refinement of this

method can be obtained with the aid of Tropical Geometry: instead of working with Newton polytopes, we can study regular subdivisions of them. So, we can cover more general families of polynomials, but at the cost of losing track of the characteristic of the ground field k.

In our case, the problem can be dealt as follows: by expanding the determinant of the generalized Vandermonde determinant with respect to the first row, we get the following expansion: $V(\underline{\mathbf{X}},\Gamma)=\sum_{i=1}^{N}(-1)^{\sigma_i}\Delta_i\mathbf{X}_1^{\gamma_i}$, For the irreducibility problem, it is enough to consider a dehomogenized version of $V(\underline{\mathbf{X}},\Gamma)$ as follows

$$V(\underline{\mathbf{X}},\Gamma)_{\mathbf{aff}} := \mathbf{X}_1^{\gamma_N}+\sum_{i=1}^{N-1}A_i\mathbf{X}_1^{\gamma_i},$$

where $A_i := (-1)^{\sigma_i}\frac{\Delta_i}{\Delta_N},\ i=1,\dots N-1$.

We can then regard $V(\underline{\mathbf{X}},\Gamma)_{\mathbf{aff}}$ as a polynomial in $K[\mathbf{X}_1]$, K being now a field containing all the A_i's, 1, $i=1,\dots,N-1$.

Given any rank one valuation: $v:K\to\mathbb{R}$, we can extend it to K^n componentwise as follows

$$\begin{array}{cccc}\mathbf{v}: & K^n & \to & \mathbb{R}^n \\ & (z_1,\dots,z_n) & \mapsto & (v(z_1),\dots,v(z_n))\end{array}.$$

The tropicalization of $V(\underline{\mathbf{X}},\Gamma)$ is then defined as

$$\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))=\overline{\mathbf{v}(\{V(\underline{\mathbf{X}},\Gamma)_{\mathbf{aff}}=0\})}\subseteq\mathbb{R}^n,$$

where the closure in the right hand side is taken with respect to the standard topology in $\mathbb{R}^n$.

It turns out (see for instance, [BG] or [EKL]) that $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ is a connected polyhedral complex of codimension 1. If $V(\underline{\mathbf{X}},\Gamma)_{\mathbf{aff}}$ is reducible over $K[\mathbf{X}_1]$, then $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ is a reducible tropical hypersurface, i.e. it can be expressed as the union of two proper tropical hypersurfaces. So, if we prove that for a special valuation $\mathbf{v}$, $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ is irreducible then $V(\underline{\mathbf{X}},\Gamma)$ will be irreducible over any field k.

Theorem 1.6. *Let $N\geq 3$. Given any field $K\supseteq k(A_1,\dots,A_{N-1})$, there exists a valuation v defined over K such that Trop$(V(\underline{\mathbf{X}},\Gamma))$ is an irreducible tropical variety if and only if:*

- $\dim(\mathcal{L}_\Gamma)\geq 2$,
- $\gcd\left(\mathbf{X}^{\gamma_i}\right)_{1\leq i\leq N}=1$, *equivalently* $\overline{\gamma}=(0,\dots,0)$.
- $d_\Gamma=1$.

This result is optimal in the following sense: it is known that $\mathrm{Trop}(\{f=0\})$ does not depend on the field, but only on the values $v(A_i)$'s.

Take for instance $f=\sum_{i=0}^{N}A_i\mathbf{Y}^i$ over a field of characteristic zero and $d_\Gamma>1$, and let $g:=\sum_{i=0}^{N}B_i\mathbf{Y}^i$ be a polynomial with the same support with coefficients in a field of characteristic $p|d_\Gamma$. Give to these polynomials valuations v and v' such that $v'(B_i)=v(A_i)$. In these conditions, we will have

$$\mathrm{Trop}(\{g=0\})=\mathrm{Trop}(\{f=0\}),$$

but $g=(\sum_{i=0}^{N}B_i^{1/p}\mathbf{Y}^{i/p})^p$ factorizes in the algebraic closure of its field of definition, hence $\mathrm{Trop}(\{f=0\})$ will always be reducible. So, the tropical criteria will not help to deduce the irreducibility of f.

The paper is organized as follows: in Section 2, we give explicit conditions on the irreducibility of the Vandermonde variety. In Section 3 we prove Theorem 1.5. We conclude by introducing some tools from Tropical Geometry and by proving Theorem 1.6 in Section 4.

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2. Bertini's Theorem and the irreducibility of the Vandermonde variety

We begin by studying the geometric irreducibility of the variety defined by $V(\underline{\mathbf{X}},\Gamma)$ in $\overline{k}^{Nn}$. In order to do this, we will apply one of the several versions of Bertini's theorem given in [Jou]. Recall that ([Jou, Definition 4.1])

Definition 2.1. A k-scheme $\mathcal{V}$ over a field k is said to be *geometrically irreducible* if $\mathcal{V}\otimes_k \overline{k}$ is an irreducible scheme.

Now we are ready to present the version of Bertini's theorem that we will use.

Theorem 2.2. [Jou, Théorème 6.3] *Let k be an infinite field, $\mathcal{V}$ a geometrically irreducible k-scheme of finite type, $\mathbb{E}^m_k$ be the affine space of dimension m, and $f:\mathcal{V}\to\mathbb{E}^m_k$ a k-morphism, i.e.*

$$\begin{array}{rccc} f & : & \mathcal{V} & \to & \mathbb{E}^m_k \\ & & z & \mapsto & \big(f_1(z),\ldots,f_m(z)\big) \end{array}$$

with $f_i\in\Gamma(\mathcal{V},\mathcal{O}_{\mathcal{V}})$. If $\mathcal{V}$ is geometrically irreducible and $\dim\big(\overline{f(\mathcal{V})}\big)\geq 2$, then for almost all $\xi\in k^{m+1}$, $f^{-1}\left(\{z\in\mathbb{E}^m_k : \xi_0+\xi_1z_1+\ldots+\xi_nz_n=0\}\right)$ is geometrically irreducible.

Definition 2.3. Let $\Gamma=\{\gamma_1,\ldots,\gamma_N\}\subset\mathbb{N}^n$, $\mathbf{Y}:=(Y_1,\ldots,Y_n)$ a set of n variables and A_i, $1\leq i\leq N$ another set of indeterminates. The *generic polynomial supported in* Γ is defined as

$$P(\mathbf{Y},\Gamma):=\sum_{i=1}^{N}A_i\,\mathbf{Y}^{\gamma_i}.$$

Proposition 2.4. *If $\dim(\mathcal{L}_\Gamma)\geq 2$ and $\gcd(\mathbf{X}^{\gamma_i})_{1\leq i\leq N}=1$, then $P(\mathbf{Y},\Gamma)$ defines an irreducible set in $\overline{k(A_1,\ldots,A_N)}^n$.*

Proof. Note that $\dim(\mathcal{L}_\Gamma)\geq 2$ implies $N\geq 3$, and moreover that there are three components of Γ that are not collinear. We pick a triple of vectors with this property which we suppose w.l.o.g. that they are γ_1, γ_2, γ_3.

In order to use Theorem 2.2, let $\mathcal{V}:=\operatorname{Spec}\left(k\left[Y_1^{\pm1},\ldots,Y_n^{\pm1}\right]\right)$ be the torus $(k^*)^n$, and set

$$\begin{array}{rccc} f & : & \mathcal{V} & \to & \mathbb{E}^{N-1}_k \\ & & z & \mapsto & \left(z^{\gamma_2-\gamma_1},z^{\gamma_3-\gamma_1},\ldots,z^{\gamma_N-\gamma_1}\right). \end{array}$$

By hypothesis, the rank of the matrix $\begin{pmatrix} \gamma_2-\gamma_1 \\ \gamma_3-\gamma_1 \\ \cdots \\ \gamma_N-\gamma_1 \end{pmatrix}$ is at least two. So, the top two by two submatrix of its Smith normal form is $\begin{vmatrix} d_1 & 0 \\ 0 & d_2 \end{vmatrix} = d_1 d_2 \neq 0$. Hence, it follows that, under a suitable monomial change of coordinates, the map f is of the form $z \mapsto (z_1^{d_1}, z_2^{d_2}, \ldots)$ and this shows that the dimension of the image of f is greater than one, so we can apply Bertini's Theorem and have that, for almost all $\xi \in \overline{k}^N$, the polynomial $Q(\xi, \mathbf{Y}) := \sum_{i=1}^N \xi_i \mathbf{Y}^{\gamma_i}$ defines an irreducible set in $(\overline{k}^*)^n$. The fact that the gcd $\left(\mathbf{Y}^{\gamma_i}\right)_{1\leq i\leq N} = 1$ implies that $Q(\xi, \mathbf{Y})$ defines an irreducible set also in $\overline{k}^n$ for almost all ξ, and hence the claim holds for $P(\mathbf{Y}, \Gamma)$.

□

Proposition 2.5. *Let $\Gamma = \left(\gamma_1, \ldots, \gamma_N\right) \subset \mathbb{N}^n$ with $N \geq 3$ and* gcd $\left(\mathbf{Y}^{\gamma_i}\right)_{1\leq i\leq N} =$ 1*. Then*

- *If $\dim(\mathcal{L}_\Gamma) = 1$, then $P(\mathbf{Y}, \Gamma)$ factorizes in $\overline{k(A_1, \ldots, A_N)}[\mathbf{Y}]$.*
- *If $\dim(\mathcal{L}_\Gamma) > 1$, then*
  - *if char(k) does not divide d_Γ, then $P(\mathbf{Y}, \Gamma)$ is absolutely irreducible.*
  - *If char$(k) = p \,|\, d_\Gamma$, then $P(\mathbf{Y}, \Gamma) = R(\mathbf{Y})^{p^r}$, with $p^r \,|\, d_\Gamma$, p^{r+1} not dividing d_Γ, and $R(\mathbf{Y}) \in \overline{k(A_1, \ldots, A_N)}[\mathbf{Y}]$ irreducible of support $\frac{1}{p^r}\Gamma$.*

Proof. If the vertices are contained in an affine line, then, via a monomial transformation, we can reduce $P(\mathbf{Y}, \Gamma)$ to a univariate polynomial, which always factorizes (due to the fact that $N > 2$) as a product of linear factors with coefficients in $\overline{k(A_1, \ldots, A_N)}$. The variety defined by this polynomial may be reducible or not, depending on the inseparability of this polynomial.

Suppose now that $\mathcal{L}_\Gamma$ has affine dimension at least two. Then we can apply the previous proposition, and conclude that the variety defined by $P(\mathbf{Y}, \Gamma)$ is irreducible over $\overline{k(A_1, \ldots, A_N)}$.

Hence, there exists an irreducible polynomial $R(\mathbf{Y}) \in \overline{k(A_1, \ldots, A_N)}[\mathbf{Y}]$ and $D \in \mathbb{N}$ such that

$$P(\mathbf{Y}, \Gamma) = R(\mathbf{Y})^D.$$

It is clear that $R(\mathbf{Y})$ cannot be a monomial. If $D = 1$ then we are done. Suppose then $D > 1$ and $p := \mathrm{char}(k)$ coprime with d_Γ. We can suppose w.l.o.g. that p does not divide the first coordinate of one of the γ_i's. But then, we have

$$0 \neq \frac{\partial P(\mathbf{Y}, \Gamma)}{\partial \mathbf{Y}_1} = \sum_{i=1}^N \gamma_{i1} A_i \, \mathbf{Y}^{\gamma_i - \mathbf{e}_1} = D R(\mathbf{Y})^{D-1} \frac{\partial R(\mathbf{Y})}{\partial \mathbf{Y}_1}. \tag{1}$$

In particular, we get that p does not divide D. As $R(\mathbf{Y})$ is not a monomial, it turns out that $\frac{\partial P(\mathbf{Y},\Gamma)}{\partial \mathbf{Y}_1}$ has at least two nonzero terms. This implies that $P(\mathbf{Y}, \Gamma)$ has at least two different monomials with positive degree on $\mathbf{Y}_1$, so the degree of $R(\mathbf{Y})$ with respect to the first variable is positive and, due to (1), the same applies to $\frac{\partial P(\mathbf{Y},\Gamma)}{\partial \mathbf{Y}_1}$.

We can then eliminate $\mathbf{Y}_1$ by computing the univariate (or classical) resultant (see [GKZ]) of the polynomials $P(\mathbf{Y}, \Gamma)$ and $\frac{\partial P(\mathbf{Y},\Gamma)}{\partial \mathbf{Y}_1}$ with respect to the first variable. This is a polynomial in $k[A_1, \ldots, A_n, \mathbf{Y}_2, \ldots, \mathbf{Y}_n]$ which must be identically

zero, as $R(\mathbf{Y})$ is a common factor of both $P(\mathbf{Y},\Gamma)$ and $\frac{\partial P(\mathbf{Y},\Gamma)}{\partial \mathbf{Y}_1}$. This is a contradiction with the fact that $A_1,\dots,A_n,\mathbf{Y}_2,\dots,\mathbf{Y}_n$ are algebraically independent.

Suppose now that $p := \mathrm{char}(k) \,|\, d_\Gamma$, let r be the maximum such that $p^r \,|\, D$, so we can write $D = p^r\, q$ with $(p,q)=1$. Write $R(\mathbf{Y}) = \sum_{j=1}^M R_j\,\mathbf{Y}^{\gamma'_j}$. We have then

$$P(\mathbf{Y},\Gamma) = \sum_{i=1}^N A_i\,\mathbf{Y}^{\gamma_i} = \left(\sum_{j=1}^M R_j\,\mathbf{Y}^{\gamma'_j}\right)^{p^r q} = \left(\sum_{j=1}^M {R_j}^{p^r}\,\mathbf{Y}^{p^r\gamma'_j}\right)^{q}.$$

From here, we deduce that p^r divides d_Γ. Moreover, after dividing all the exponents by p^r we get

$$P_r(\mathbf{Y},\Gamma) := \sum_{i=1}^N A_i\,\mathbf{Y}^{\frac{\gamma_i}{p^r}} = \left(\sum_{j=1}^M {R_j}^{p^r}\,\mathbf{Y}^{\gamma'_j}\right)^{q}.$$

It turns out that $P_r(\mathbf{Y},\Gamma) = P(\mathbf{Y},\frac{1}{p^r}\Gamma)$. An argument like above over $\frac{1}{p^r}\Gamma$ shows that q cannot be different than one if the A_i are algebraically independent. This completes the proof. $\square$

As in the introduction, for $\ell = 1,\dots,N$ we set

$$\Delta_\ell(\underline{\mathbf{X}},\Gamma) := \det\left({\mathbf{X}_i}^{\gamma_j}\right)_{\substack{2\le i\le N,\, 1\le j\le N\\ j\neq \ell}},$$

i.e. Δ_ℓ is the minor obtained by deleting the first row and ℓ-th column in the Generalized Vandermonde matrix.

Theorem 2.6. *For any index ℓ_0, the family $\{\Delta_\ell/\Delta_{\ell_0} \;:\; \ell = 1,\dots N, \ell\neq\ell_0\}$ is algebraically independent over any field k.*

Proof. Suppose without loss of generality that $\ell_0 = N$. It is easy to see that Δ_N does not define the zero function on $\overline{k}^{(n-1)N}$ even if $\mathrm{char}(k) > 0$.

Let $\mathcal{V}$ be the Zariski image of the rational map:

$$\begin{array}{ccc} \overline{k}^{n(N-1)} & \longrightarrow & \overline{k}^{n(N-1)+(N-1)} \\ (\mathbf{X}_2,\dots,\mathbf{X}_N) & \mapsto & \left(\mathbf{X}_2,\dots,\mathbf{X}_N,\frac{\Delta_1}{\Delta_N},\dots,\frac{\Delta_{N-1}}{\Delta_N}\right) \end{array}$$

It is clear that this is a birational map between $\overline{k}^{n(N-1)}$ and $\mathcal{V}$. Let $\mathcal{I}$ be the ideal of $\mathcal{V}$ in $k[\mathbf{X}_2,\dots,\mathbf{X}_N,a_1,\dots,a_{N-1}]$. $\mathcal{I}$ is a prime ideal that contains the polynomials $\Delta_i - a_i\Delta_N, i<N$ and -by Cramer's rule- $f(\mathbf{X}_\ell) = \mathbf{X}_\ell^{\gamma_N} + \sum_{i=1}^{N-1} a_i\mathbf{X}_\ell^{\gamma_i}$, $2\le\ell\le N$. Let $\mathbf{a} = \{a_1,\dots,a_{N-1}\}$. By construction, the field of rational functions of $\mathcal{V}$ is isomorphic to the field of fractions of the integer domain

$$\mathbb{L} = \mathrm{Frac}\left(\frac{k[\mathbf{X}_2,\dots,\mathbf{X}_N,\mathbf{a}]}{\mathcal{I}}\right) \simeq k(\mathbf{X}_2,\dots,\mathbf{X}_N).$$

In particular, $(\mathbf{X}_2,\dots,\mathbf{X}_N)$ is a transcendence basis of $k\subset\mathbb{L}$ and the dimension of $\mathbb{L}$ is $n(N-1)$. For each index $2\le\ell\le N$, choose one variable X_{ℓ,j_ℓ} appearing in $f(\mathbf{X}_l)$. Denote by $\mathbf{X}_0 = \{\mathbf{X}_2,\dots,\mathbf{X}_N\}\setminus\{X_{2,j_2},\dots,X_{N,j_N}\}$ the remaining variables $X_{i,j}$ not chosen. As an element in $\mathbb{L}$, X_{ℓ,j_ℓ} is algebraic over $k(\mathbf{X}_0,\mathbf{a})$, because $f(X_{\ell,j_\ell})\in\mathcal{I}$. So $\mathbb{L}$ itself is an algebraic extension of $k(\mathbf{X}_0,\mathbf{a})$. The set $\{\mathbf{X}_0,\mathbf{a}\}$ is of cardinal $(n-1)(N-1)+(N-1) = n(N-1)$. So it is a transcendence basis of $\mathbb{L}$ over k. In particular, this means that the set $\{a_1,\dots a_{N-1}\}$ is algebraically

independent over $\mathbb{L}$, and hence $\{\frac{\Delta_1}{\Delta_N}, \dots, \frac{\Delta_{N-1}}{\Delta_N}\}$ is algebraically independent over k. $\square$

3. Proof of Theorem 1.5

With all the preliminaries given in Section 2, we can prove the main theorem. It is clear that, if any of the three conditions in the statement of Theorem 1.5 fail to hold, then $V(\underline{\mathbf{X}}, \Gamma)$ factorizes.

Suppose then that these conditions are satisfied. By developing $V(\underline{\mathbf{X}}, \Gamma)$ as a polynomial in the variables indexed by $\mathbf{X}_1$, we have the following

$$V(\underline{\mathbf{X}}, \Gamma) = \sum_{i=1}^{N} (-1)^{\sigma_i} \Delta_i \mathbf{X}_1^{\gamma_i},$$

with $\sigma_i \in \{0, 1\}$. Hence, we can regard $V(\underline{\mathbf{X}}, \Gamma)$ as the polynomial $P(\mathbf{X}_1, \Gamma)$ specialized under $A_i \mapsto (-1)^{\sigma_i} \Delta_i$.

As the family $((-1)^{\sigma_i} \Delta_i / \Delta_N)_{1 \le i \le N-1}$ is algebraically independent (due to Theorem 2.6), the polynomial $\mathbf{X}_1^{\gamma_N} + \sum_{i=1}^{N-1} (-1)^{\sigma_i} \Delta_i / \Delta_N \mathbf{X}_1^{\gamma_i}$ is generic among the polynomials of support Γ, monic in γ_1. This means that, for almost every $t_{ij}, 2 \le i \le N, 1 \le j \le N$, the set of zeroes of $V(\underline{\mathbf{X}}, \Gamma)$ in $\overline{k}^n$ after setting $X_{ij} \mapsto t_{ij}$ is irreducible (by Proposition 2.4).

As a consequence of this, we get that the set of zeroes of $V(\underline{\mathbf{X}}, \Gamma)$ is irreducible in $\overline{k(\mathbf{X}_2, \dots, \mathbf{X}_N)}$, and hence -as in Proposition 2.5- $V(\underline{\mathbf{X}}, \Gamma)$ must be the power of an irreducible polynomial. By using again Proposition 2.5 and our hypothesis, we conclude that $V(\underline{\mathbf{X}}, \Gamma)$ is irreducible in $\overline{k(\mathbf{X}_2, \dots, \mathbf{X}_N)}[X_1]$.

In order to show irreducibility in $\overline{k}[\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N]$, we argue as follows: if it does factorize in this ring, then it must have an irreducible factor depending only on $\mathbf{X}_2, \dots, \mathbf{X}_N$. It cannot be a monomial by the second hypothesis. So, it is a proper factor of positive degree in -we can assume w.l.o.g.- $\mathbf{X}_2$ and degree zero in $\mathbf{X}_1$. We then have

$$V(\underline{\mathbf{X}}, \Gamma) = p(\mathbf{X}_2, \dots, \mathbf{X}_N) q(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N),$$

with $\deg_{\mathbf{X}_2}(p) > 0$. By making the change of coordinates $\mathbf{X}_2 \leftrightarrow \mathbf{X}_1$, we get

$$-V(\underline{\mathbf{X}}, \Gamma) = p(\mathbf{X}_1, \mathbf{X}_3, \dots, \mathbf{X}_N) q(\mathbf{X}_2, \mathbf{X}_1, \dots, \mathbf{X}_N). \tag{2}$$

If $\deg_{\mathbf{X}_2}(q(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N)) > 0$, then minus the right hand side of (2) is a factorization of $V(\underline{\mathbf{X}}, \Gamma)$ with two factors os positive degree in $\mathbf{X}_1$, a contradiction with the irreducibility over $\overline{k(\mathbf{X}_2, \dots, \mathbf{X}_N)}[\mathbf{X}_1]$. So, we must actually have

$$V(\underline{\mathbf{X}}, \Gamma) = p(\mathbf{X}_2, \dots, \mathbf{X}_N) q(\mathbf{X}_1, \mathbf{X}_3, \dots, \mathbf{X}_N).$$

But now, if we set $\mathbf{X}_1 = \mathbf{X}_2$ in $V(\underline{\mathbf{X}}, \Gamma)$, we get

$$0 = p(\mathbf{X}_1, \dots, \mathbf{X}_N) q(\mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_N),$$

a contradiction with the fact that neither p nor q are zero. Hence, the irreducibility of $V(\underline{\mathbf{X}}, \Gamma)$ follows.

4. The tropical approach

As in the introduction, the expression $V(\underline{\mathbf{X}},\Gamma)=\sum_{i=1}^{N}(-1)^{\sigma_i}\Delta_i\mathbf{X}_1^{\gamma_i}$ corresponds to the development of the generalized Vandermonde determinant with respect to the first row of its defining matrix. We dehomogenize again this polynomial as

$$\mathbf{X}_1^{\gamma_N}+\sum_{i=1}^{N-1}A_i\mathbf{X}_1^{\gamma_i},$$

where the A_i's are algebraically independent over k by Theorem 2.6.

Given any rank one valuation: $v:\overline{k(A_1,\ldots,A_{N-1})}\to\mathbb{R}$, we define $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$, the tropicalization of $V(\underline{\mathbf{X}},\Gamma)$, as the closure of $\{V(\underline{\mathbf{X}},\Gamma)=0\}\subset\overline{k(A_1,\ldots,A_{N-1})}^n$ under this valuation.

Let $\Lambda\subset\mathbb{R}^n$ be the convex hull of Γ. The values $v(A_i)$ define a regular subdivision $Subdiv(\Lambda)$ that is combinatorially dual to $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ ([Mik, Proposition 3.11]). In particular, by the duality, the vertices of $Subdiv(\Lambda)$ correspond to the connected components of $\mathbb{R}^n\setminus\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$, the edges of $Subdiv(\Lambda)$ correspond to the facets of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ and the two-dimensional polytopes of $Subdiv(\Lambda)$ correspond to the ridges of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$. There are more cells, but we will focus only on these.

Every facet F of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ has associated a *multiplicity* as follows: let e be the corresponding dual edge of F in $Subdiv(\Lambda)$. The multiplicity of F is defined as $\#(e\cap\mathbb{Z}^n)-1$, the integer length of e. With this definition, the *balancing condition* on the ridges of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ holds: given any such ridge R, let $F_1,\ldots,F_r$ be the facets containing R in their boundary, m_i be the multiplicity of F_i and v_i the primitive integer normal vector to the affine hyperplane generated by F_i chosen with a compatible orientation. Then:

$$\sum_{i=1}^{r}m_iv_i=0$$

We refer to [Mik] or [TS] for more background on this subject. We will use the balancing condition to show the irreducibility of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$.

Proof of Theorem 1.6.

Proof. If one of the hypotheses of 1.6 is not fulfilled, then, it is easy to find a field k where $V(\underline{\mathbf{X}},\Gamma)$ factors, and hence $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ will be reducible. Suppose then that the three conditions hold and let k be any field.

Consider $\mathbf{X}_1^{\gamma_N}+\sum_{i=1}^{N-1}A_i\mathbf{X}_1^{\gamma_i}$, where A_i are rational functions in $\{\mathbf{X}_2,\ldots,\mathbf{X}_N\}$, algebraically independent over k. Hence, any function $v:\{A_1,\ldots,A_{N-1}\}\to\mathbb{R}$, can be extended to a valuation

$$\mathbf{v}:\overline{k(\mathbf{X}_2,\ldots,\mathbf{X}_N)}\to\mathbb{R}.$$

For our proof, we need a function that induces a regular triangulation of Γ. We may take, for instance, any appropriate generic infinitesimal perturbation of the standard paraboloid lifting function:

$$v(A_i)=\sum_{j=1}^{n}((\gamma_{ij}+\epsilon_{ij})^2-\gamma_{Nj}^2).$$

This function induces a Delaunay triangulation of the set of exponents $\{\gamma_1,\ldots,\gamma_N\}$ (See [GR]).

The tropicalization of $V(\underline{\mathbf{X}},\Gamma)$ under this valuation is combinatorially dual to this triangulation. So in particular, the ridges of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ are always the intersection of three facets, because their dual cell is always a triangle. Moreover, for any such intersection, the compatible primitive vectors involved in the balancing condition are two by two linearly independent.

Suppose that $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma)) = H_1 \cup H_2$. Let F_1 be a facet of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ and suppose that $F_1 \subseteq H_1$. We want to prove that $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma)) \subseteq H_1$ as sets of points. Let R be any ridge incident to F_1 and F_2, F_3 the other two facets incident to R. Let m_i be the weight of F_i as a facet of H_1, so $m_i = 0$ if and only if F_i is not a facet of H_1. Let v_i be the compatible primitive vector associated to F_i and R. Since $F_1 \in H_1$, its weight must be a positive integer, $m_1 > 0$.

From the balancing condition, we have that $m_1v_1 + m_2v_2 + m_3v_3 = 0$. Since $m_1 > 0$ and v_1, v_2, v_3 are pairwise linearly independent vectors, it must happen that $m_2 > 0, m_3 > 0$. That is, F_2, F_3 have positive weight, so they belong to H_1 as sets of points. To sum up, for any facet F of $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ belonging to H_1 it happens that the facets that are ridge-connected to F also belong to H_1. Now, since Γ is not contained in a line, it is known that $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ is ridge-connected, that is, every two facets can be connected by a path of facets such that any two of them that are consecutive have a common ridge.

We can then conclude by induction by showing that $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma)) = H_1$ as subsets of $\mathbb{R}^n$. In particular, $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma))$ cannot factorize as the union of two different tropical hypersurfaces, set-theoretically.

However, it could still happen that $\mathrm{Trop}(V(\underline{\mathbf{X}},\Gamma)) = H_1 \cup H_2$ with $H_1 = H_2$ as sets, but being different only by the multiplicities of the facets. In that case, let m_i^1, m_i^2 be the multiplicity of F_i as a facet of H_1 and H_2 respectively. Then, it is easy to check that $m_i^1/m_i^2 = p/q$ is a rational constant that does not depend on the facet. Thus, there are integer positive numbers m_i^0 such that $m_i^1 = k_1m_i^0$, $m_i^2 = k_2m_i^0$, where $k_1, k_2 \in \mathbb{Z}_{>0}$ are constants not depending on the facet i. Hence, the multiplicity of F_i as a facet of H is $(k_1+k_2)m_i^0$, and this imply that every facet has a multiplicity which is a multiple of $k_1 + k_2 \geq 2$. By duality, every edge of $Subdiv(\Lambda)$ will have an integer length multiple of $(k_1 + k_2)$. It follows that d_Γ is a multiple of $k_1 + k_2$, which contradicts the hypotheses. $\square$

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